SMA Problem Corner (February, 2021)

Geometry

Instruction: Provide answers together with their proofs

Warm up!

Let ABC be a right-angled triangle with right angle at A. Let D and E be two points on BC such that B, C, D, E are all distinct. Let the circumcircles of ABD and ACE meet again at F, and let the circumcircles of ABE and ACD meet again at G, and let the circumcircles of BDG and CDF meet again at H. How many cyclic quadrilaterals are there?

Level 1

Let A, B, C, D, E, F, G, H be distinct points on a plane such that $BCDE, ABEF, CDAF, BCFG, DABG, CDGH, ABCH$ and $DAHE$ are all cyclic. It is given that no three points are collinear and $ABCD$ is not cyclic. How many cyclic quadrilaterals are there?

Level 2

Now include the circumcentres of the cyclic quadrilaterals found in *Level 1*. How many *new* cyclic quadrilaterals are there?

Level 3

Let τ be a collection of points on the plane such that each circle in *Level 2* passes through at least one point in τ. Find the minimum number of points τ can have.

Bonus!!!

Find all pairs of circles in *Level 1* whose radical axis and radial axis are the radial axis and radical axis of another pair of circles in *Level 1* respectively.

(**Radial axis** of two circles is the line passing through their centres. Can check the net for more detailed explanation for **radical axis**.)

Proposed by: Pionaj

Number Theory

Prove or disprove

If a, b and c are pairwise coprime positive odd integers. Then there exist unique positive integers x, y and z such that;

i) $a = \gcd(2x - 1, 2y, 2z + 1)$,

ii) $b = \gcd(2y - 1, 2z, 2x + 1)$,

iii) $c = \gcd(2z - 1, 2x, 2y + 1)$,

iv) $abc \geq \max(x, y, z)$.

Proposed by: Pionaj