SMA Problem Corner (June, 2021)

Number Theory

Instruction: Provide answers together with their proofs

Warm up!

If $x_1, x_2, x_3, y_1, y_2, y_3$ are integers satisfying

$$\begin{cases} x_1y_2 - x_2y_1 = 1, \\ x_2y_3 - x_3y_2 = 1, \\ x_3y_1 - x_1y_3 = 1. \end{cases}$$

Show that $x_1 + x_2 + x_3 = 0$.

Level 1

Let $n \geq 2$. Suppose there exist non-zero integers $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ such that $x_iy_{i+1} - x_{i+1}y_i = 1$ for $i = 1,2, \cdots, n-1$, and $x_ny_1 - x_1y_n = 1$. Show that

$$\frac{1}{x_1x_2} + \frac{1}{x_2x_3} + \cdots + \frac{1}{x_{n-1}x_n} + \frac{1}{x_nx_1} = 0.$$

Level 2

Find all $n \geq 2$ such that there exist integers $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ satisfying $x_iy_{i+1} - x_{i+1}y_i = 1$ for $i = 1,2, \cdots, n-1$, and $x_ny_1 - x_1y_n = 1$.

Level 3

Prove or Disprove.

Let $n \geq 2$. Suppose there exist integers $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ such that $x_iy_{i+1} - x_{i+1}y_i = 1$ for $i = 1,2, \cdots, n-1$, and $x_ny_1 - x_1y_n = 1$.

Must there exist distinct $i_1, i_2, \cdots, i_k$ such that $x_{i_1} + x_{i_2} + \cdots + x_{i_k} = 0$?

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